

THE DISTINGUISHING NUMBERS OF MERGED JOHNSON GRAPHS

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ABSTRACT. In present article, we determine the distinguishing number of the merged Johnson graphs which are generalization of both the Kneser graphs and of the Johnson graphs.

1. INTRODUCTION

The *distinguishing number* of a graph G is the minimum number of colors for which there exists an assignment of colors to the vertices of G such that the identity is the only color-preserving automorphism of G . Generally, for a permutation group Γ acting on X , the *distinguishing number* of Γ is the minimum number of cells of a partition π of X satisfying that the identity is the only element of Γ fixing each cell of π . Albertson and Collins first introduced distinguishing number of a graph [3] and there have been many interesting results on the distinguishing numbers of graphs and permutation groups in last few years [1–5, 7, 8, 10, 12].

Here we consider a class of graphs based on the Johnson graphs. For positive integers k, n such that $1 \leq k \leq \frac{n}{2}$, the *Johnson graph* $J(n, k)$ has vertices given by the k -subsets of $[n] = \{1, 2, \dots, n\}$ and there exists an edge between two vertices if and only if their intersection has size $k - 1$. Given a nonempty subset $I \subseteq \{1, 2, \dots, k\}$, the *merged Johnson graph* $J(n, k)_I$ is the union of the distance i graphs $J(n, k)_i$ of $J(n, k)$ for all i , namely, two k -subsets are adjacent in $J(n, k)_I$ if and only if their intersection has $k - i$ elements for some $i \in I$. The merged Johnson graphs $J(n, k)_I$ include many interesting graphs such as the Johnson graph $J(n, k) = J(n, k)_{\{1\}}$ and the Kneser graph $K(n, k) = J(n, k)_{\{k\}}$.

In [1], M.O. Albertson and D.L. Boutin determined the distinguishing number of the Kneser graphs. The aim of the present article is to determine the distinguishing number of the merged Johnson graphs.

The outline of this paper is as follows. In section 2, we review some preliminaries regarding the distinguishing numbers and the merged Johnson graphs. In section 3, we find Theorem 3.2 which addresses the distinguishing numbers of the merged Johnson graphs. We also prove lemmas which are used for a proof of the main theorem. At last, we provide a proof of Theorem 3.2 in section 4.

2. PRELIMINARIES

For a given graph G , a coloring $f : V(G) \rightarrow \{1, 2, \dots, r\}$ is said to be *r -distinguishing* if the identity is the only graph automorphism ϕ satisfying $f(\phi(v)) = f(v)$ for all $v \in V(G)$. This means that the distinguishing coloring is a symmetry-breaking coloring of G . The

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distinguishing number, denoted by $\text{Dist}(G)$, is the minimum r that G has an r -distinguishing coloring. One can easily see that $\text{Dist}(G) = \text{Dist}(G^c)$ where G^c is the complement of G . If G is an *asymmetric graph*, the identity is the only automorphism of G , then $\text{Dist}(G) = 1$. In fact, the converse is also true.

For a graph G and for a subset $S \subseteq V(G)$, a coloring $f : S \rightarrow \{1, 2, \dots, s\}$ is said to be *s-distinguishing* if for any graph automorphism $\phi \in \text{Aut}(G)$ fixing S set-wisely and if $f(\phi(v)) = f(v)$ for all $v \in S$ then ϕ fixes all elements of S . In this case, ϕ does not need to fix other vertices outside of the given set S . If there exists an s -distinguishing coloring for S , the set S is called an *s-distinguishable set*.

For a graph G , a subset $S \subseteq V(G)$ is called a *determining set* if the identity is the only automorphism fixing every element of S . The *determining number* of G , denoted by $\text{Det}(G)$, is the minimum number t that G has a determining set of the cardinality t . It is also true that $\text{Det}(G) = \text{Det}(G^c)$. For example, for any $n \geq 3$, $\text{Det}(C_n) = 2$ because the set of two adjacent vertices is a determining set of C_n and there is no determining set of C_n of the cardinality 1. Note that if $S \subseteq V(G)$ is a determining set, then any subset $T \subseteq V(G)$ containing S is also a determining set. The determining sets provide a useful tool for finding the distinguishing number of G as stated in the following theorem.

Theorem 2.1. ([1]) *For a given graph G , G has an r -distinguishable determining set if and only if G has a $(r + 1)$ -distinguishing coloring.*

Consequently, we find the following corollary which will be used in the proof of our results.

Corollary 2.2. *For a given graph G , if there is a determining set S of G such that the induced subgraph of S is asymmetric then $\text{Dist}(G) = 1$ or 2.*

Proof. For a graph automorphism ϕ of G fixing S set-wisely, the restriction of ϕ on the induced subgraph $\langle S \rangle$ is a graph automorphism of $\langle S \rangle$. Since $\langle S \rangle$ is asymmetric, the coloring $f(v) = 1$ for all $v \in S$ is a 1-distinguishing. Furthermore, since S is a determining set, G has 2-distinguishing coloring by Theorem 2.1. \square

Using Theorem 2.1, M.O. Albertson and D.L. Boutin determined the distinguishing number of the Kneser graphs as follows.

Proposition 2.3. ([1]) *For any integers $n > k \geq 2$ with $n > 2k$, $\text{Dist}(K(n, k)) = 2$ except $(n, k) = (5, 2)$.*

Note that the Kneser graph $K(5, 2)$ is isomorphic to the Petersen graph and its distinguishing number is 3. $K(n, 1)$ is isomorphic to the complete graph and so $\text{Dist}(K(n, 1)) = n$ for any integer n .

For a graph G , the distinguishing number $\text{Dist}(G)$ equal to the distinguishing number of $\text{Aut}(G)$ which acts on the vertex set $V(G)$. Hence we have the following lemma. The proof is straightforward and we omit it.

Lemma 2.4. *Let G_1 and G_2 be two graphs having the same vertex set V .*

- (1) *If $\text{Aut}(G_1)$ is a subgroup of $\text{Aut}(G_2)$ as acting groups on V , then $\text{Dist}(G_1) \leq \text{Dist}(G_2)$.*
- (2) *If $\text{Aut}(G_1) = \text{Aut}(G_2)$ as acting groups on V , then $\text{Dist}(G_1) = \text{Dist}(G_2)$.*

3. THE DISTINGUISHING NUMBERS OF THE MERGED JOHNSON GRAPHS

Let Ω be the set of all k -subsets of $[n]$. The action of S_n on $[n]$ naturally induces an action of S_n on Ω . Then the Johnson graph $J(n, k)$ is an orbital graph which corresponds to the orbital $\{(M, N) \in \Omega^2 \mid |M \cap N| = k - 1\}$, that is, the orbit of S_n on Ω^2 . The *distance i graph* $J(n, k)_i$ of $J(n, k)$ is an orbital graph corresponding to the orbital

$$\Gamma_i = \{(M, N) \in \Omega^2 \mid |M \cap N| = k - i\}$$

(see [6] for orbital graphs).

For a merged Johnson graph $J = J(n, k)_I$, the complementation of each k -sets $M \rightarrow M^c$ induces an isomorphism from $J(n, k)_I$ to $J(n, n - k)_I$. So we may assume that $k \leq n/2$. For a merged Johnson graph $J = J(n, k)_I$ with $1 \leq k \leq \frac{n}{2}$, if $I = \emptyset$ or $\{1, 2, \dots, k\}$ then J is the null or complete graph and so $\text{Aut}(J) = S_d$, where $d = \binom{n}{k}$. Thus, we further assume that $k \geq 2$ and $\emptyset \subsetneq I \subsetneq \{1, 2, \dots, k\}$. For notational simplicity, let $I' = I \setminus \{k\}$, and for any integer t , let $t - I = \{t - i \mid i \in I\}$ and $t - I' = \{t - i \mid i \in I'\}$. We also denote $I'' = k - I'$, and let $e = \frac{1}{2}\binom{n}{n/2}$.

In [9], G. Jones found the automorphism groups of the merged Johnson graphs as follows.

Theorem 3.1. ([9]) *Let $J = J(n, k)_I$, where $2 \leq k \leq \frac{n}{2}$ and $\emptyset \subsetneq I \subsetneq \{1, 2, \dots, k\}$ and let $A = \text{Aut}(J)$.*

- (1) *If $2 \leq k < \frac{n-1}{2}$, and $J \neq J(12, 4)_I$ with $I = \{1, 3\}$ or $\{2, 4\}$, then $A = S_n$ with orbitals $\Gamma_0, \Gamma_1, \dots, \Gamma_k \subset \Omega^2$.*
- (2) *If $J = J(12, 4)_I$ with $I = \{1, 3\}$ or $\{2, 4\}$, then $A = O_{10}^{-1}(2)$ with orbitals $\Gamma_0, \Gamma_1 \cup \Gamma_3, \Gamma_2 \cup \Gamma_4$.*
- (3) *If $k = \frac{n-1}{2}$ and $I \neq k + 1 - I$, then $A = S_n$ with orbitals $\Gamma_0, \Gamma_1, \dots, \Gamma_k \subset \Omega^2$.*
- (4) *If $k = \frac{n-1}{2}$ and $I = k + 1 - I$, then $A = S_{n+1}$ with orbitals Γ_0 and $\Gamma_i \cup \Gamma_{k+1-i}$ for all $i = 1, 2, \dots, \lfloor \frac{k+1}{2} \rfloor$.*
- (5) *If $k = \frac{n}{2}$ and $I \neq \{k\}$ nor $\{1, 2, \dots, k-1\}$, and $I' \neq I''$, then $A = S_2 \times S_n$ with orbitals $\Gamma_0, \Gamma_1, \dots, \Gamma_k \subset \Omega^2$.*
- (6) *If $k = \frac{n}{2}$ and $I \neq \{k\}$ nor $\{1, 2, \dots, k-1\}$, and $I' = I''$, then $A = S_2^e : S_n$ with orbitals Γ_0 and $\Gamma_i \cup \Gamma_{k-i}$ for all $i = 1, 2, \dots, \lfloor \frac{k}{2} \rfloor$ and Γ_k .*
- (7) *If $k = \frac{n}{2}$ and $I = \{k\}$ or $\{1, 2, \dots, k-1\}$, then $A = S_2^e : S_e = S_2 \wr S_e$ with orbitals $\Gamma_0, \Gamma_1 \cup \dots \cup \Gamma_{k-1}$ and Γ_k .*

To understand the automorphism group S_{n+1} of $J(n, \frac{n-1}{2})_I$ with $I = k + 1 - I$, let $\tilde{[n]} = [n] \cup \{\infty\}$ and let Ψ be the set of equipartitions of $\tilde{[n]}$, by which we mean the unordered partitions $\{P_1, P_2\}$ of $\tilde{[n]}$ satisfying $|P_1| = |P_2| = \frac{n+1}{2}$. There is a bijection $\phi : \Omega \rightarrow \Psi$, sending each M to $\{M \cup \{\infty\}, [n] - M\}$. Note that its inverse sends an equipartition $\{P_1, P_2\}$ to $P_i \setminus \{\infty\}$, where i is chosen so that $\infty \in P_i$. The natural action of S_{n+1} on $\tilde{[n]}$ induces an action of S_{n+1} on Ω . By the condition $I = k + 1 - I$, one can see that this action induces an automorphism group of $J(n, \frac{n-1}{2})_I$ (For a detail information, see the paper [9]). The next theorem is the main theorem of this paper.

Theorem 3.2. *Let $J = J(n, k)_I$, where $1 \leq k \leq \frac{n}{2}$ and $\emptyset \subsetneq I \subsetneq \{1, 2, \dots, k\}$.*

- (1) *If $k = 1$ then $\text{Dist}(J) = n$.*

- (2) If $(n, k) \neq (5, 2)$ and $J \neq J(n, \frac{n}{2})_I$ with $I = \{\frac{n}{2}\}, \{1, 2, \dots, \frac{n}{2} - 1\}$ or $I' = I''$, then $\text{Dist}(J) = 2$.
- (3) If $J = J(5, 2)_I$ with $I = \{1\}$ or $\{2\}$; or $J = J(n, \frac{n}{2})_I$ satisfying $I' = I''$ and I is neither $\{\frac{n}{2}\}$ nor $\{1, 2, \dots, \frac{n}{2} - 1\}$, then $\text{Dist}(J) = 3$.
- (4) If $J = J(n, \frac{n}{2})_I$ with $I = \{\frac{n}{2}\}$ or $\{1, 2, \dots, \frac{n}{2} - 1\}$, then $\text{Dist}(J) = \lceil \frac{1 + \sqrt{1 + 4(\frac{n}{2})}}{2} \rceil$.

Corollary 3.3. *Let $G = J(n, k)$ be the Johnson graph. Then*

- (1) if $k = 1$, then $\text{Dist}(G) = n$;
- (2) if $(n, k) \neq (4, 2)$ nor $(5, 2)$, then $\text{Dist}(G) = 2$ and
- (3) if $G = J(4, 2)$ or $J(5, 2)$, then $\text{Dist}(J) = 3$.

We will prove Theorem 3.2 in the next section. For the rest of the section, we will prove lemmas which will be used in the proof of Theorem 3.2. For a set $X = \{1, 2, 3, \dots, n\}$ and for any permutation ϕ of X , ϕ can be represented by (i_1, i_2, \dots, i_n) , where for any $j = 1, 2, \dots, n$, $\phi(j) = i_j$. Throughout the rest of the paper, we use the above representation of permutations. For any permutation $\phi = (i_1, i_2, \dots, i_n)$ and for any $k, \ell \in [n]$ with $1 \leq k \leq \frac{n}{2}$, let V_ℓ^ϕ be the k -subset $\{i_\ell, i_{\ell+1}, \dots, i_{\ell+k-1}\}$, where the subscripts are considered as their residue classes modulo n . For our convenience, if ϕ is the identity, we denote V_ℓ^ϕ simply by V_ℓ . For any $i = 1, 2, \dots, n$, let τ_i be the transposition of X exchanging i and $i+1$.

Lemma 3.4. *Let $J = J(12, 4)_I$ with $I = \{1, 3\}$. For any permutation ϕ of X , if an automorphism Ψ of J fixes all vertices in $S = \{V_j^\phi, V_j^{\phi\tau_1}, V_j^{\phi\tau_2} \mid j = 1, 2, \dots, 12\}$, then Ψ also fixes $V_j^{\phi\tau_i}$ for all $i, j = 1, 2, \dots, 12$.*

Proof. Without any loss of generality, we may assume that ϕ is the identity. Note that $S = \{V_j \mid j = 1, 2, \dots, 12\} \cup \{V_2^{\tau_1}, V_{10}^{\tau_1}\} \cup \{V_3^{\tau_2}, V_{11}^{\tau_2}\}$, where $V_2^{\tau_1} = \{1, 3, 4, 5\}$, $V_{10}^{\tau_1} = \{10, 11, 12, 2\}$ and $V_3^{\tau_2} = \{2, 4, 5, 6\}$, $V_{11}^{\tau_2} = \{11, 12, 1, 3\}$.

Let Ψ be an automorphism of J fixing all vertices in S . Note that $\{V_j^{\tau_3} \mid j = 1, 2, \dots, 12\} - S = \{V_4^{\tau_3}, V_{12}^{\tau_3}\}$ because $\tau_3 = (1, 2, 4, 3, 5, 6, \dots, 12)$, where $V_4^{\tau_3} = \{3, 5, 6, 7\}$ and $V_{12}^{\tau_3} = \{12, 1, 2, 4\}$. Let A and B be the sets of all vertices in S which are adjacent to $V_4^{\tau_3}$ and $V_{12}^{\tau_3}$, respectively. Then we have

$$A = \{V_1, V_3, V_4, V_5, V_7, V_{12}, V_{11}^{\tau_2}\} \quad \text{and} \quad B = \{V_1, V_3, V_4, V_9, V_{11}, V_{12}\}.$$

Let V be a vertex whose all adjacent vertices in S is A . For the first case, assume that $|V \cap V_1| = 3$. If $V \cap V_1 = \{1, 2, 3\}$, then $4, 5, 6 \notin V$ and $7 \in V$ because V is adjacent to both V_3 and V_4 but not to V_2 . In this case, V is adjacent to V_6 , a contradiction. If $V \cap V_1 = \{1, 2, 4\}$, then $5, 6, 7 \notin V$ and $8 \in V$, and hence V is adjacent to V_6 , a contradiction. Similarly, one can show that $V \cap V_1$ is neither $\{1, 3, 4\}$ nor $\{2, 3, 4\}$. Therefore, we have $|V \cap V_1| = 1$.

By considering the fact that V is adjacent to $V_1, V_3, V_4, V_5, V_7, V_{12}$ but not to $V_2, V_6, V_8, V_9, V_{10}, V_{11}$, one can show that $V = \{1, 6, 9, 11\}, \{2, 5, 10, 11\}$ or $\{3, 5, 6, 7\}$. Since V is not adjacent to $V_2^{\tau_1} = \{1, 3, 4, 5\}$, V is $V_4^{\tau_3} = \{3, 5, 6, 7\}$. This implies that Ψ also fixes $V_4^{\tau_3}$.

Let V' be a vertex whose all adjacent vertices in S is B . By considering the fact that V' is adjacent to $V_1, V_3, V_4, V_9, V_{11}, V_{12}$ but not to $V_2, V_5, V_6, V_7, V_8, V_{10}$, one can show that $V' = \{1, 6, 8, 10\}, \{2, 5, 8, 9\}$ or $\{12, 1, 2, 4\}$. Since V' is not adjacent to $V_2^{\tau_1} = \{1, 3, 4,$

5}, V' is $V_{12}^{\tau_3} = \{12, 1, 2, 4\}$. This implies that Ψ fixes $V_{12}^{\tau_3}$. Up to now, we showed that Ψ fixes $V_j^{\tau_3}$ for all $j = 1, 2, \dots, 12$.

Since Ψ fixes all elements in $\{V_j, V_j^{\tau_2}, V_j^{\tau_3} \mid j = 1, 2, \dots, 12\}$, one can show that Ψ fixes $V_j^{\tau_4}$ for all $j = 1, 2, \dots, 12$ by a similar way. Continuing the similar process, one can show that Ψ fixes $V_j^{\tau_i}$ for all $i, j = 1, 2, \dots, 12$. \square

Lemma 3.5. *Let $J = J(12, 4)_I$ with $I = \{1, 3\}$. For any permutation ϕ of X , if an automorphism Ψ of J fixes all vertices in $S_1 = \{V_j^\phi, V_j^{\phi\tau_i} \mid i, j = 1, 2, \dots, 12\}$, then Ψ also fixes $V_j^{\phi\tau_i\tau_k}$ for all $i, j, k = 1, 2, \dots, 12$.*

Proof. Without any loss of generality, assume that ϕ is the identity. Let Ψ be an automorphism of J fixing all vertices in S_1 . Note that $\{V_j^{\tau_1\tau_2} \mid j = 1, 2, \dots, 12\} - S_1 = \{V_3^{\tau_1\tau_2}, V_{11}^{\tau_1\tau_2}\}$ because $\tau_1\tau_2 = (2, 3, 1, 4, 5, \dots, 12)$, where $V_3^{\tau_1\tau_2} = \{1, 4, 5, 6\}$ and $V_{11}^{\tau_1\tau_2} = \{11, 12, 2, 3\}$. Let C and D be the sets of all vertices in S_1 which are adjacent to $V_3^{\tau_1\tau_2}$ and $V_{11}^{\tau_1\tau_2}$, respectively. Then we have

$$\{V_3, V_4, V_6, V_{10}, V_{11}, V_{12}, V_2^{\tau_1}, V_3^{\tau_2}\} \subset C \quad \text{and} \quad \{V_3, V_8, V_{11}, V_{12}, V_2^{\tau_1}, V_3^{\tau_2}\} \subset D.$$

Let V be a vertex whose all adjacent vertices in S_1 is C . By considering the fact that V is adjacent to $V_3, V_4, V_6, V_{10}, V_{11}, V_{12}$ but not to $V_1, V_2, V_5, V_7, V_8, V_9$, we have that $V = \{6, 8, 9, 12\}, \{1, 4, 5, 6\}$ or $\{2, 4, 9, 10\}$. Since V is adjacent to both $V_2^{\tau_1} = \{1, 3, 4, 5\}$ and $V_3^{\tau_2} = \{2, 4, 5, 6\}$, V is $V_3^{\tau_1\tau_2} = \{1, 4, 5, 6\}$. This implies that Ψ also fixes $V_4^{\tau_3}$.

Let V' be a vertex whose all adjacent vertices in S_1 is D . By considering the fact that V' is adjacent to V_3, V_8, V_{11}, V_{12} but not to $V_1, V_2, V_4, V_5, V_6, V_7, V_9, V_{10}$, one can show that $V' = \{6, 7, 10, 12\}, \{2, 4, 7, 8\}$ or $\{11, 12, 2, 3\}$. Since V' is adjacent to both $V_2^{\tau_1} = \{1, 3, 4, 5\}$ and $V_3^{\tau_2} = \{2, 4, 5, 6\}$, V' is $V_{11}^{\tau_1\tau_2} = \{11, 12, 2, 3\}$. This implies that Ψ fixes $V_{11}^{\tau_1\tau_2}$. Therefore Ψ fixes $V_j^{\tau_1\tau_2}$ for all $j = 1, 2, \dots, 12$.

Since Ψ fixes all vertices in $\{V_j^\phi, V_j^{\phi\tau_1}, V_j^{\phi\tau_2} \mid j = 1, 2, \dots, 12\}$ with $\phi = \tau_1$, Ψ also fixes $V_j^{\tau_1\tau_i}$ for all $i, j = 1, 2, \dots, 12$ by Lemma 3.4. By a similar way, one can show that Ψ fixes $V_j^{\tau_i\tau_k}$ for all $i, j, k = 1, 2, \dots, 12$. \square

Lemma 3.6. *Let $J = J(12, 4)_I$ with $I = \{1, 3\}$. Then $S = \{V_j, V_j^{\tau_1}, V_j^{\tau_2} \mid j = 1, 2, \dots, 12\}$ is a determining set.*

Proof. Let Ψ be an automorphism of J fixing all vertices in $S = \{V_j, V_j^{\tau_1}, V_j^{\tau_2} \mid j = 1, 2, \dots, 12\}$. By Lemma 3.4, Ψ fixes $V_j^{\tau_i}$ for all $i, j = 1, 2, \dots, 12$. Furthermore, Ψ fixes $V_j^{\tau_i\tau_k}$ for all $i, j, k = 1, 2, \dots, 12$ by Lemma 3.5. By applying Lemma 3.5 again with $\phi = \tau_i$, one can show that Ψ fixes $V_j^{\tau_i\tau_k\tau_\ell}$ for all $i, j, k, \ell = 1, 2, \dots, 12$. Continuing the similar process, one can show that Ψ fixes $V_j^{\tau_{i_1}\tau_{i_2}\dots\tau_{i_t}}$ for any positive integer t and for all $1 \leq i_1, i_2, \dots, i_t, j \leq 12$. Since $\{\tau_i \mid i = 1, 2, \dots, 12\}$ generates symmetric group on $X = \{1, 2, 3, \dots, 12\}$, Ψ fixes all permutation of X , i.e., S is a determining set. \square

Let J be a merged Johnson graph $J(2m, m)_I$ with $I \subseteq \{1, 2, \dots, m\}$. For any $v \in V(G)$, let \bar{v} be the vertex $[2m] - v$ for our convenience.

Lemma 3.7. *Let $J = J(n, k)_I$, where $1 \leq k \leq \frac{n}{2}$ and $\emptyset \subsetneq I \subsetneq \{1, 2, \dots, k\}$.*

- (1) *For $(n, k) = (2m + 1, m)$ with $m \geq 3$ and $I = \{1, m\}$, $S_1 = \{V_1, V_2, \dots, V_{m+2}\}$ is a determining set.*

- (2) For $(n, k) = (2m, m)$ with $m \geq 3$ and $I = \{1\}$, $S_2 = \{V_1, V_2, \dots, V_{2m}\} \cup \{\{1, 2, \dots, m-2, m, m+2\}\}$ is a determining set.

Proof. (1) For any automorphism ϕ of J as a permutation on vertices of J , let ϕ' be a corresponding permutation of $[n] \cup \{\infty\}$. Let ϕ be an automorphism of J fixing all elements in S_1 . Since ϕ fixes V_1 , ϕ' fixes $\{1, 2, \dots, m, \infty\}$ set-wisely or ϕ' sends $\{1, 2, \dots, m, \infty\}$ to $\{m+1, m+2, \dots, 2m+1\}$ set-wisely.

Case 1: ϕ' fixes $\{1, 2, \dots, m, \infty\}$ set-wisely.

Since ϕ fixes V_2 and ϕ' fixes $\{1, 2, \dots, m, \infty\}$ set-wisely, ϕ' also fixes $\{2, 3, \dots, m+1, \infty\}$ set-wisely. This implies that ϕ' fixes both 1 and $m+1$. Using the fact that ϕ fixes all elements in S_1 , one can see that ϕ' fixes all elements in $[2m+1] = [2m+1] \cup \{\infty\}$, namely, ϕ is the identity element.

Case 2: ϕ' sends $\{1, 2, \dots, m, \infty\}$ to $\{m+1, m+2, \dots, 2m+1\}$ set-wisely.

Since ϕ fixes V_2 and ϕ' sends $\{1, 2, \dots, m, \infty\}$ to $\{m+1, m+2, \dots, 2m+1\}$ set-wisely, ϕ' also sends $\{2, 3, \dots, m+1, \infty\}$ to $\{m+2, m+3, \dots, 2m+1, 1\}$ set-wisely. By the similar way, one can show that for any $i = 1, 2, \dots, m+2$, ϕ' sends $\{i, i+1, \dots, m+i-1, \infty\}$ to $[\tilde{n}] - \{i, i+1, \dots, m+i-1\}$ set-wisely.

Let $\phi'(\infty) = a$. Then a is contained to $\{m+1, m+2, \dots, 2m+1\}$ and hence ϕ' can not send $\{a-m+1, a-m+2, \dots, a, \infty\}$ to $[2m+1] - \{a-m+1, a-m+2, \dots, a\}$ set-wisely, which is a contradiction.

Therefore $S_1 = \{V_1, V_2, \dots, V_{m+2}\}$ is a determining set.

- (2) Let α be the automorphism which sends v to \bar{v} for all $v \in V(G)$. Then the order of α is 2 and $\text{Aut}(J) \cong \langle \alpha \rangle \times S_n$. Note that for any automorphism ψ of J , ψ is either an automorphism induced by a permutation of $[n]$ or a product of α and an automorphism induced by a permutation of $[n]$.

Let ψ be an automorphism of J fixing all elements in S_2 . Since ψ fixes V_1 , ψ is an automorphism induced by a permutation of $[n]$ fixing $\{1, 2, \dots, m\}$ set-wisely or ψ is a product of α and an automorphism induced by a permutation of $[n]$ sending $\{1, 2, \dots, m\}$ to $\{m+1, m+2, \dots, 2m\}$ set-wisely.

If ψ is an automorphism induced by a permutation ψ' of $[n]$ fixing $\{1, 2, \dots, m\}$ set-wise, then ψ' also fixes $\{i, i+1, \dots, i+m-1\}$ set-wise for all $i = 1, 2, \dots, 2m$ because ψ fixes V_i for all $i = 1, 2, \dots, 2m$. This implies that ψ is the identity. Hence we can assume that ψ is a product of α and an automorphism induced by a permutation ψ'' of $[n]$ sending $\{1, 2, \dots, m\}$ to $\{m+1, m+2, \dots, 2m\}$ set-wisely. Since ψ fixes V_2 , ψ'' also sends $\{2, 3, \dots, m+1\}$ to $\{m+2, m+3, \dots, 2m, 1\}$ set-wisely. This implies that ψ'' exchanges 1 and $m+1$. By a similar way, one can show that ψ'' exchanges i and $m+i$ for any $i = 1, 2, \dots, m$. But in this case, ψ does not fix $\{1, 2, \dots, m-2, m, m+2\}$. Therefore, S_2 is a determining set. \square

Lemma 3.8. For $m \geq 4$, let $J = J(2m, m)_I$ with $I = \{1, m-1\}$. Then, $\text{Dist}(J) > 2$.

Proof. For any $v \in V(J)$, let β_v be the automorphism of $V(J)$ exchanging v and \bar{v} and fixing all other vertices. Let $f : V(J) \rightarrow \{1, 2\}$ be a coloring. If there exists $u \in V(J)$ such that $f(u) = f(\bar{u})$ then β_u is a color-preserving automorphism, and hence f is not

2-distinguishing. Assume that for all $v \in V(J)$, $f(v)$ and $f(\bar{v})$ are distinct. Let ϕ be an automorphism of J induced by a non-identity permutation of $[2m]$. Let

$$\psi = \left(\prod_{\{u, \bar{u}\}, f(u) \neq f(\phi(u))} \beta_u \right) \phi.$$

Then ψ is a color-preserving automorphism. Therefore, there is no 2-distinguishing coloring, and hence $\text{Dist}(J) > 2$. \square

For $n = 2m$ with $m \geq 4$, let Φ be the set of equipartitions of $[n]$. Note that the size of Φ is $\frac{1}{2} \binom{n}{m}$. The natural action of S_n on $[n]$ induces an action of S_n on Φ . Let α be a permutation satisfying that for all non-identity permutation $\beta \in S_n$, the number of equipartitions fixed by α in Φ is greater than or equal to the number of equipartitions fixed by β in Φ . Suppose that there exist a permutation $\gamma \in S_n$ and $i_1, i_2, \dots, i_t \in [n]$ with $t \geq 3$ such that $\gamma(i_j) = i_{j+1}$ for all $j = 1, 2, \dots, t-1$ and $\gamma(i_t) = i_1$. Let $\tilde{\gamma} \in S_n$ be a permutation that $\tilde{\gamma}(i_2) = i_1$, $\tilde{\gamma}(i_t) = i_3$ and $\tilde{\gamma}(\ell) = \gamma(\ell)$ for all $\ell \in [n] \setminus \{i_2, i_t\}$. Then, all equipartitions in Φ fixed by γ are also fixed by $\tilde{\gamma}$. This implies that α is a product of disjoint transpositions. For two permutations $\gamma_1, \gamma_2 \in S_n$, suppose that there exist $i_1, i_2, \dots, i_6 \in [n]$ such that

$$\begin{aligned} \gamma_1(i_1) &= i_2, \gamma_1(i_2) = i_1, \gamma_1(i_3) = i_4, \gamma_1(i_4) = i_3, \gamma_1(i_5) = i_5, \gamma_1(i_6) = i_6, \\ \gamma_2(i_1) &= i_2, \gamma_2(i_2) = i_1, \gamma_2(i_3) = i_3, \gamma_2(i_4) = i_4, \gamma_2(i_5) = i_5, \gamma_2(i_6) = i_6, \end{aligned}$$

and for all $j \in [n] \setminus \{i_1, i_2, \dots, i_6\}$, $\gamma_1(j) = \gamma_2(j)$. Then, one can check that all equipartitions fixed by γ_1 are also fixed by γ_2 . This implies that α is a transposition or a product of m disjoint transpositions. Note that the number of equipartitions fixed by a transposition is $\binom{2m-2}{m-2}$ and the number of equipartitions fixed by a product of m disjoint transpositions is 2^{m-1} if m is odd; $2^{m-1} + \binom{m}{m/2}$ if m is even. Since for $m \geq 4$, $\binom{2m-2}{m-2} > 2^{m-1} + \binom{m}{m/2}$, α is a transposition. Therefore for any non-identity permutation $\beta \in S_n$, the number of equipartitions fixed by β is at most $\binom{2m-2}{m-2}$.

Lemma 3.9. *For $n = 2m$ with $m \geq 4$, let Φ be the set of equipartitions of $[n]$. Then there is a 3-coloring $c : \Phi \rightarrow \{B, R, Y\}$ of Φ such that only identity permutation in S_n preserves all colors under the induced action of S_n on Φ .*

Proof. Give a random coloring on Φ with three colors $\{B, R, Y\}$. For any non-identity permutation β of $[2m]$, let A_β be the event that β preserves colors of all equipartitions of $[2m]$. Note that the number of equipartitions fixed by β is at most $\binom{2m-2}{m-2}$. Namely, the number of equipartitions which are not fixed by β is at least

$$|\Phi| - \binom{2m-2}{m-2} = \frac{1}{2} \binom{2m}{m} - \binom{2m-2}{m-2} = \frac{m}{m-1} \binom{2m-2}{m-2}.$$

For any orbit O of β whose size is t with $t \geq 2$ under the action of S_{2m} on Φ , the probability that β preserves colors of all equipartitions in O is 3^{-t+1} , which is less than $3^{-\frac{t}{2}}$. Hence, we have

$$\Pr(A_\beta) \leq 3^{-\frac{m}{2(m-1)} \binom{2m-2}{m-2}}.$$

Therefore,

$$Pr(\bigcup_{\beta \in S_n \setminus \{id\}} A_\beta) \leq \sum_{\beta \in S_n \setminus \{id\}} Pr(A_\beta) < ((2m)! - 1)3^{-\frac{m}{2(m-1)}\binom{2m-2}{m-2}} \leq (2m)!3^{-\frac{m}{2(m-1)}\binom{2m-2}{m-2}},$$

where id is the identity permutation of $[n]$. For $n = 8$, the number $n!3^{-\frac{m}{2(m-1)}\binom{2m-2}{m-2}}$ is $\frac{4480}{3^8}$, and it is less than 1. Furthermore, for any $m \geq 4$,

$$\frac{(2m+2)!3^{-\frac{m+1}{2m}\binom{2m}{m-1}}}{(2m)!3^{-\frac{m}{2(m-1)}\binom{2m-2}{m-2}}} = \frac{(2m+2)(2m+1)}{3^{\frac{3m-2}{2(m-1)}\binom{2m-2}{m-2}}} < \frac{(2m+2)(2m+1)}{3^{\frac{3}{2}\binom{2m-2}{m-2}}} < 1$$

because

$$\begin{aligned} 3^{\frac{3}{2}\binom{2m-2}{m-2}} &= 3^{\frac{3(2m-2)(2m-3)\cdots(m+1)}{2\cdot(m-2)!}} = 3^{\frac{(m-1)(2m-3)(2m-5)(2m-6)\cdots(m+1)}{(m-3)(m-4)\cdots 5\cdot 4}} \\ &> 3^{(m-1)(2m-3)(2m-5)} > (2m+2)(2m+1) \end{aligned}$$

for any $m \geq 4$. This implies that for any $m \geq 4$,

$$Pr(\bigcup_{\beta \in S_n \setminus \{id\}} A_\beta) < 1$$

and hence there exists a 3-coloring $c : \Phi \rightarrow \{B, R, Y\}$ of Φ such that the identity permutation in S_{2m} is the only color-preserving permutation under the induced action of S_{2m} on Φ . \square

4. A PROOF OF THE MAIN THEOREM

In this section, we prove Theorem 3.2 which is the main result in this paper. For a graph G and for a $v \in V(G)$, let $N(v)$ be the set of all vertices adjacent to v . For any positive integer i , let D_i be the set of all vertices whose degrees are i .

Let $J = J(n, k)_I$, where $1 \leq k \leq \frac{n}{2}$ and $\emptyset \subsetneq I \subsetneq \{1, 2, \dots, k\}$. If $k = 1$ then the graph J is a complete graph K_n , and hence its distinguishing number is n . Assume that $k > 1$.

Case 1: $(n, k) \neq (5, 2)$ and $2 \leq k < \frac{n-1}{2}$, and $J \neq J(12, 4)_I$ with $I = \{1, 3\}$ or $\{2, 4\}$.

In this case, $\text{Aut}(J) = \text{Aut}(K(n, k))$ as an acting group on the vertex set, and hence $\text{Dist}(J) = \text{Dist}(K(n, k)) = 2$ by Proposition 2.3 and Lemma 2.4.

Case 2: $J = J(12, 4)_I$ with $I = \{1, 3\}$ or $\{2, 4\}$.

Assume that $J = J(12, 4)_I$ with $I = \{1, 3\}$. Let $S_1 = \{V_j, V_j^{\tau_1}, V_j^{\tau_2} \mid j = 1, 2, \dots, 12\} \cup \{X_1 = \{1, 3, 5, 7\}\}$ and let H_1 be a subgraph of J induced by S_1 as illustrated in Figure 1. Then S_1 is a determining set by Lemma 3.6. Let ψ_1 be an automorphism of H_1 . Note that the order of H_1 is 17 and

$$\begin{aligned} D_5 &= \{V_4, V_5, V_8, V_9\}, \quad D_6 = \{V_1, V_6, V_7, V_{12}, V_{11}^{\tau_2}, X_1\} \\ D_7 &= \{V_3, V_2^{\tau_1}, V_{10}^{\tau_1}\}, \quad D_8 = \{V_2, V_{10}, V_3^{\tau_2}\}, \quad D_9 = \{V_{11}\} \end{aligned}$$

Since $D_9 = \{V_{11}\}$, $\psi_1(V_{11}) = V_{11}$. The fact $N(V_{11}) \cap D_5 = \{V_8\}$ implies that $\psi_1(V_8) = V_8$. Since $N(V_8) \cap D_5 = \{V_5, V_9\}$, ψ_1 fixes V_4 . By the fact that $N(V_4) \cap N(V_8) \cap D_5 = \{V_5\}$, ψ_1 fixes both V_5 and V_9 . Note that $N(V_8) \cap N(V_{11}) = \{V_{11}^{\tau_2}\}$. So ψ_1 fixes $V_{11}^{\tau_2}$. The fact $N(V_4) \cap N(V_8) \cap D_6 = \{V_7\}$ and $N(V_5) \cap N(V_9) \cap D_6 = \{V_6\}$ implies that ψ_1 fixes both V_6 and V_7 . Furthermore, ψ_1 fixes $X_1 = \{1, 3, 5, 7\}$ because $N(V_6) \cap N(V_7) = \{X_1\}$. Since

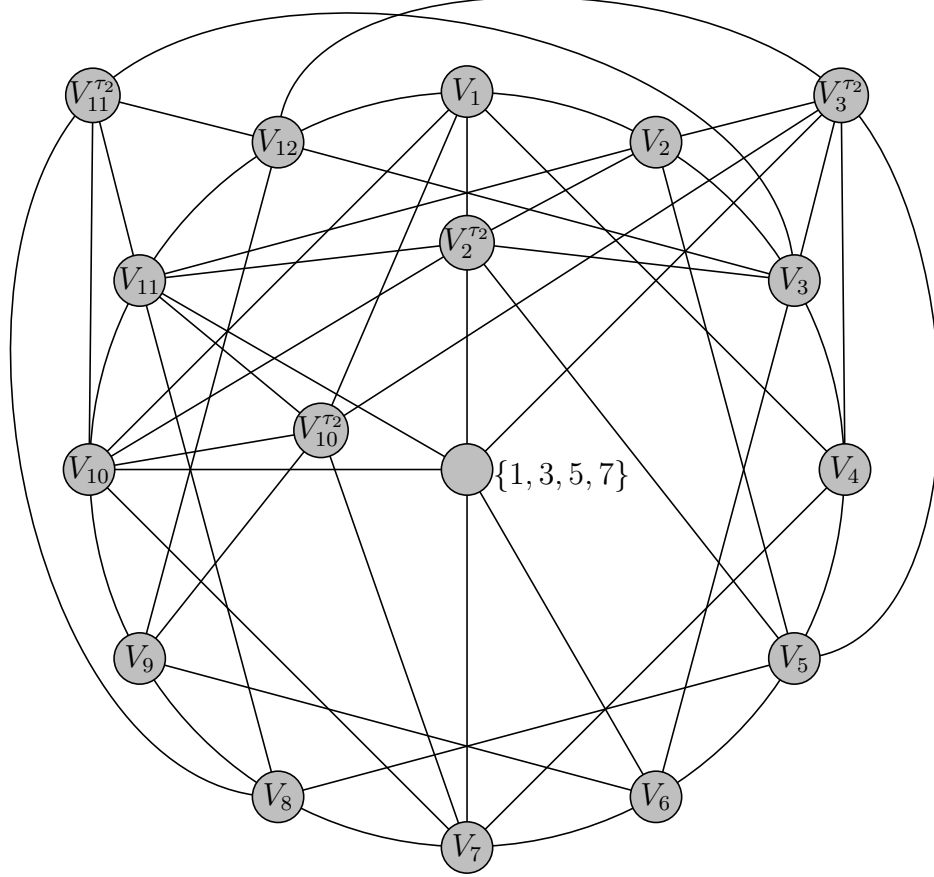


FIGURE 1. .

$N(V_7) \cap N(V_{11}) \cap D_7 = \{V_{10}^{\tau_1}\}$ and $N(V_7) \cap N(V_{11}) \cap D_8 = \{V_{10}\}$, ψ_1 also fixes both V_{10} and $V_{10}^{\tau_1}$. Note that $N(V_4) \cap N(V_6) \cap D_7 = \{V_3\}$ and $N(V_4) \cap N(V_6) \cap D_8 = \{V_3^{\tau_2}\}$. This implies that $\psi_1(V_3) = V_3$ and $\psi_1(V_3^{\tau_2}) = V_3^{\tau_2}$. By the fact $N(V_5) \cap N(V_{11}) = \{V_2^{\tau_1}\}$ and $N(V_9) \cap N(V_{11}^{\tau_2}) \cap D_6 = \{V_{12}^{\tau_2}\}$, ψ_1 fixes both $V_2^{\tau_1}$ and $V_{12}^{\tau_2}$. Up to now, we know that ψ_1 fixes all vertices in $V(H)$ except V_1 and V_2 . Since the degree of V_1 is 6 and that of V_2 is 8, ψ_1 also fixes both V_1 and V_2 . Therefore, ψ_1 is the identity, which implies that H is an asymmetric graph. By Corollary 2.2, we have $\text{Dist}(J) = 2$.

For any $J_1 = J(12, 4)_I$ with $I = \{2, 4\}$, $\text{Aut}(J_1) = \text{Aut}(J)$. Therefore, $\text{Dist}(J_1) = \text{Dist}(J) = 2$.

Case 3: $J = J(5, 2)_I$ with $I = \{1\}$ or $\{2\}$.

Since $J(5, 2)_{\{2\}}$ is the Kneser graph $K(5, 2)$ and $J(5, 2)_{\{1\}}$ is its complement, we have $\text{Dist}(J) = 3$ by Proposition 2.3.

Case 4: $k = \frac{n-1}{2}$ and $I \neq k + 1 - I$.

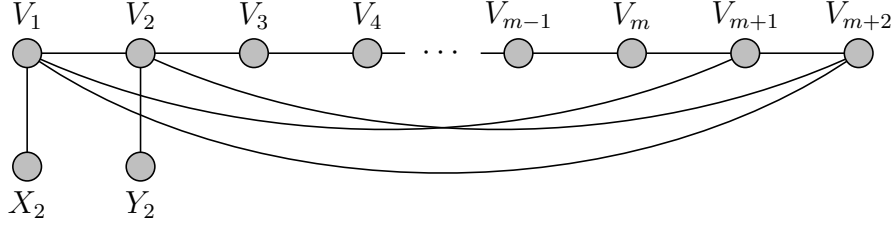


FIGURE 2. .

If $n = 5$ then $I = \{1\}$ or $\{2\}$, which means that $I = k + 1 - I$. Hence we have $n \geq 7$. In this case, since $\text{Aut}(J) = \text{Aut}(K(n, \frac{n-1}{2}))$ as an acting group on the vertex set, one can find that $\text{Dist}(J) = \text{Dist}(K(n, k)) = 2$ by Proposition 2.3 and Lemma 2.4.

Case 5: $k = \frac{n-1}{2}$ and $I = k + 1 - I$.

Let $J = J(2m + 1, m)_{\{1, m\}}$ with $m \geq 3$. Let

$$S_2 = \{V_1, V_2, \dots, V_{m+2}\} \cup \{X_2, Y_2\},$$

where $X_2 = \{1, 2, \dots, m-2, m, m+2\}$ and $Y_2 = \{2, 3, \dots, m-1, m+1, m+3\}$; and let H_2 be a subgraph of J induced by S_2 as depicted in Figure 2. Then, S_2 is a determining set of J by Lemma 3.7(1). Let ψ_2 be an automorphism of J . Note that

$$D_1 = \{X_2, Y_2\}, \quad D_2 = \{V_3, V_4, \dots, V_m\}, \quad D_3 = \{V_{m+1}, V_{m+2}\}, \quad D_4 = \{V_1, V_2\}.$$

Since V_1 is the only vertex adjacent to all vertices in D_3 , $\psi_2(V_1) = V_1$. This implies that ψ_2 fixes V_2, X_2 and Y_2 because $D_4 = \{V_1, V_2\}$; and X_2 and Y_2 are only adjacent to V_1 and V_2 , respectively. By the fact that V_{m+2} is the only vertex adjacent to all vertices in D_4 , $\psi_2(V_{m+2}) = V_{m+2}$. This implies that ψ_2 also fixes V_{m+1} because $D_3 = \{V_{m+1}, V_{m+2}\}$. Since $N(V_2) \cap D_2 = \{V_3\}$, $\psi_2(V_3) = V_3$. By a similar way, one can show that ψ_2 fixes all vertices in H_2 , and hence ψ_2 is the identity. This means that H_2 is asymmetric. By Corollary 2.2, we have $\text{Dist}(J) = 2$.

For any $J_2 = J(2m + 1, m)_I$ satisfying $I = k + 1 - I$, $\text{Aut}(J_2) = \text{Aut}(J)$. Therefore $\text{Dist}(J_2) = \text{Dist}(J) = 2$.

Case 6: $k = \frac{n}{2}$ and I is neither $\{k\}$ nor $\{1, 2, \dots, k-1\}$, and $I' \neq I''$.

Let $J = J(2m, m)_{\{1\}}$ with $m \geq 3$. Let

$$S_3 = \{V_1, V_2, \dots, V_{2m}\} \cup \{X_3, Y_3, Z_3\},$$

where $X_3 = \{1, 2, \dots, m-2, m, m+2\}$, $Y_3 = \{2, 3, \dots, m-1, m+1, m+3\}$ and $Z_3 = \{4, 5, \dots, m+1, m+3, m+5\}$; and let H_3 be a subgraph of J induced by S_3 as shown in Figure 3. Then, S_3 is a determining set of J by Lemma 3.7(2). Let ψ_3 be an automorphism of J . Note that

$$D_1 = \{X_3, Y_3, Z_3\}, \quad D_3 = \{V_1, V_2, V_4\} \quad \text{and} \quad D_2 = V(H_3) - (D_1 \cup D_3).$$

By a similar way with Cases 2 and 5, one can show that ψ_3 fixes all vertices in H_3 , and hence ψ_3 is the identity. So H_3 is an asymmetric graph. By Corollary 2.2, we have $\text{Dist}(J) = 2$.

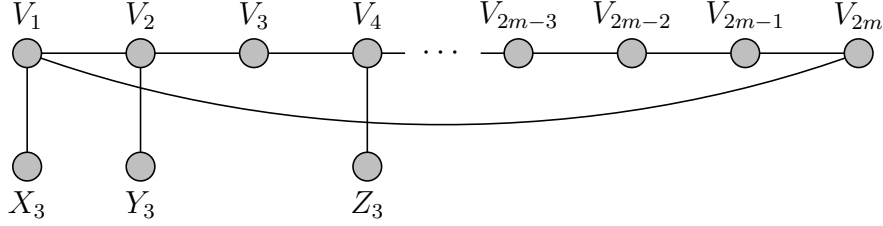


FIGURE 3. .

For any $J_3 = J(2m, m)_I$ satisfying $I \neq \{k\}$ nor $\{1, 2, \dots, k-1\}$, and $I' \neq I''$, $\text{Aut}(J_3) = \text{Aut}(J)$. Therefore, we find $\text{Dist}(J_3) = \text{Dist}(J) = 2$.

Case 7: $k = \frac{n}{2}$, $I' = I''$ and I is neither $\{k\}$ nor $\{1, 2, \dots, k-1\}$.

Note that for $n \leq 6$, this case can not occur. Hence assume that $n \geq 8$. Let $J = J(2m, m)_{\{1, m-1\}}$ with $m \geq 4$. By Lemma 3.8, we have $\text{Dist}(J) \geq 3$. Let $f : V(J) \rightarrow \{1, 2, 3\}$ be a random coloring satisfying that for any $u \in V(J)$, $f(u)$ and $f(\bar{u})$ are distinct. Note that for any $u \in V(J)$,

$$\Pr(\{f(u), f(\bar{u})\} = \{1, 2\}) = \Pr(\{f(u), f(\bar{u})\} = \{1, 3\}) = \Pr(\{f(u), f(\bar{u})\} = \{2, 3\}) = \frac{1}{3}.$$

Let Φ be the set of equipartitions of $[n]$. Using a random coloring $f : V(J) \rightarrow \{1, 2, 3\}$, we can define the coloring $\tilde{f} : \Phi \rightarrow \{B, R, Y\}$ as follows: for any $\{u, \bar{u}\} \in \Phi$, $\tilde{f}(\{u, \bar{u}\}) = B$ if $\{f(u), f(\bar{u})\} = \{1, 2\}$; $\tilde{f}(\{u, \bar{u}\}) = R$ if $\{f(u), f(\bar{u})\} = \{1, 3\}$; and $\tilde{f}(\{u, \bar{u}\}) = Y$ if $\{f(u), f(\bar{u})\} = \{2, 3\}$. Then we can consider \tilde{f} as a random 3-coloring of Φ . By Lemma 3.9, there is a random 3-coloring $f : V(J) \rightarrow \{1, 2, 3\}$ such that only identity permutation in S_n preserves all colors of equipartitions in its corresponding random 3-coloring $\tilde{f} : \Phi \rightarrow \{B, R, Y\}$ under the induced action of S_n on Φ . This implies that there exists a 3-distinguishing coloring $f : V(J) \rightarrow \{1, 2, 3\}$. Therefore, $\text{Dist}(J) \leq 3$, and hence $\text{Dist}(J) = 3$.

For any $J_4 = J(2m, m)_I$ satisfying $I' = I''$ and I is neither $\{k\}$ nor $\{1, 2, \dots, k-1\}$, $\text{Aut}(J_4) = \text{Aut}(J)$. Therefore, $\text{Dist}(J_4) = \text{Dist}(J) = 3$.

Case 8: $k = \frac{n}{2}$ and $I = \{k\}$ or $\{1, 2, \dots, k-1\}$.

Let $J = J(2m, m)_{\{m\}}$. Then J is composed of $\frac{\binom{2m}{m}}{2}$ components which are isomorphic to K_2 . Note that a coloring $f : V(J) \rightarrow \{1, 2, \dots, r\}$ is an r -distinguishing if and only if for any vertex $u \in V(J)$, $f(u)$ and $f(\bar{u})$ are distinct and for any two vertex $v, w \in V(J)$ contained to different components, $\{f(v), f(\bar{v})\} \neq \{f(w), f(\bar{w})\}$. Hence $\text{Dist}(J)$ is the smallest integer r such that $\binom{r}{2} \geq \frac{\binom{2m}{m}}{2}$. Therefore, $\text{Dist}(J) = \lceil \frac{1 + \sqrt{1 + 4\binom{2m}{m}}}{2} \rceil$.

For any $J_5 = J(2m, m)_I$ with $I = \{1, 2, \dots, m-1\}$, J_5 is the complement of J . Hence, $\text{Dist}(J_5) = \text{Dist}(J) = \lceil \frac{1 + \sqrt{1 + 4\binom{2m}{m}}}{2} \rceil$.

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